ON POSITIVE SOLUTIONS OF THE HOMOGENEOUS HAMMERSTEIN INTEGRAL EQUATION

YU. KH. ESHKABILOV, F.H. HAYDAROV

Abstract. In this paper the existence and uniqueness positive fixed points of the one nonlinear integral operator are discussed. We prove that existence finite positive solutions of the integral equation of Hammerstein type. Obtained results applied to study Gibbs measures for models on a Cayley tree.

Key words. integral equation of Hammerstein type, fixed point of operator, Gibbs measure, Cayley tree.

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1. Introduction

It is well known that integral equations have wide applications in engineering, mechanics, physics, economics, optimization, vehicular traffic, biology, queuing theory and so on (see [15], [16], [19], [2], [17]). The theory of integral equations is rapidly developing with the help of tools in functional analysis, topology and fixed point theory. Therefore, many different methods are used to obtain the solution of the nonlinear integral equation. Moreover, some methods can be found in Refs. [6], [10], [1], [21], [4], [7], [5], [3], to discuss and obtain the solution of Hammerstein integral equation. In [7] J.Appell and A.S. Kalitvin used fixed point methods and methods of nonlinear spectral theory to obtain the solution of integral equations of Hammerstein or Uryson type. The existence of positive solutions of abstract integral equations of Hammerstein type is discussed in [5]. In [3] M.A. Abdou, M.M. El-Borai and M.M. El-Kojok the existence and uniqueness solution of the nonlinear integral equation of Hammerstein type with discontinuous kernel are discussed.

This present paper, we study solvability homogeneous integral equation of Hammerstein type. An integral equation of the form

$$\int_0^1 K(t,u)\Psi(t,f(u))du = f(t) \tag{1.1}$$

is called the homogeneous Hammerstein integral equation, where $K(t,u)$ is continuous real-valued function defined on $0 \leq t \leq 0,\ \ 0 \leq u \leq 1,\ \ \Psi : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function and $f(t)$ is unknown function from $C[0,1]$.

Let $\Psi(t,z), \frac{\partial}{\partial z}\Psi(t,z)$ be continuous and bounded for $t \in [0,1]$ and for all z. Then [20] the Hammerstein integral equation (1.1) has a solution. Assume that $\Psi(t,z)$ is a bounded continuous function for $t \in [0,1]$ and $z \in \mathbb{R}$. In this case, also the Hammerstein integral equation (1.1) has [18] a solution. For the necessary details of this theorem and for more

results on the Hammerstein integral equation, we refer to Petryshyn and Fitzpatrik [22], Browder [9], Brezis and Browder [8].

Recently, in [11] consider the case $\Psi(t,z)=\Psi(z)$. Let $\Psi(z)$ is monotone left-continuous function on $[0,+\infty)$ and $lim_{x\to 0}\frac{\Psi(z)}{z}=+\infty,\ \ lim_{x\to +\infty}\frac{\Psi(z)}{z}=0$. Then [11] the integral equation of Hammerstein type (1.1) has a solution.

In this work, we will consider the following integral equation of Hammerstein type (i.e. in (1.1) $\Psi(t,z)=\Psi(z)=z^{\vartheta}$):

$$\int_0^1 K(t,u)f^{\vartheta}(u)du=f(t),\ \ \vartheta>1 \tag{1.2}$$

on the $C[0,1]$, where $K(t,u)$ is strictly positive continuous function.

By the Theorem 44.8 from [17] follows the existence of nontrivial positive solution of the Hammerstein equation (1.2). We study the problem of existence finite number positive solutions of the integral equation of Hammerstein type (1.2).

Consider the nonlinear operator R_α on the cone of positive continuous functions on $[0,1]$:

$$\left(R_\alpha f\right)(t)=\left(\frac{\int_0^1 K(t,u)f(u)du}{\int_0^1 K(0,u)f(u)du}\right)^{\alpha}, \tag{1.3}$$

where $K(t,u)$ is given in the integral equation of Hammerstein type (1.1) and $\alpha>0$. Operator of the form (1.3) arising in the theory of Gibbs measures (see [13], [12], [25]). Positive fixed points of the operator $R_k,\ k\in\mathbb{N}$ and they numbers is very important to study Gibbs measures for models on a Cayley tree.

In [13], in the case $\alpha=1$ the uniqueness positive fixed points of the nonlinear operator R_α (1.3) is proved. In [12], in the case $\alpha=k\in\mathbb{N}, k>1$ for the nonlinear operator R_α was proved the existence of positive fixed point and the existence Gibbs measure for some mathematical models on a Cayley tree.

The aim of this work is to study the existence finite number positive solutions of the Hammerstein equation (1.2) on the space of continuous functions on $[0,1]$. The plan of this paper is as follows. In the second section using properties of Hammerstein equation (1.2) we reduce some statements on the positive fixed point of the operator R_α. In the third section we construct the strictly positive continuous kernel $K(t,u)$ such that, the corresponding Hammerstein equation (1.2) has $n\in\mathbb{N}$ positive solutions. In the fourth section obtained results for the operator R_α applied to study Gibbs measures for models on a Cayley tree.

2. Existence and uniqueness of positive fixed points of the operator R_α

In this section we study the existence and the uniqueness positive fixed points of the nonlinear operator R_α (1.3). Put

$$C^+[0,1]=\{f\in C[0,1]: f(x)\geq 0\},\ \ C_0^+[0,1]=C^+[0,1]\setminus\{\theta\equiv 0\}.$$

Well then the set $C^+[0,1]$ is the cone of positive continuous functions on $[0,1]$.

We define the Hammerstein operator H_ϑ on $C[0,1]$ by the equality

$$H_\vartheta f(t)=\int_0^1 K(t,u)f^{\vartheta}(u)du=f(t),\ \ \vartheta>1.$$

Clearly, that by the Theorem 44.8 from [17] we obtained

Theorem 1. *Let $\vartheta > 1$. The equation*

$$H_\vartheta f = f \tag{2.1}$$

has at least one solution in $C_0^+[0,1]$.

Put

$$\mathcal{M}_0 = \left\{ f \in C^+[0,1] : f(0) = 1 \right\}.$$

Lemma 1. *Let $\alpha > 1$. The equation*

$$R_\alpha f = f, \quad f \in C_0^+[0,1] \tag{2.2}$$

has a positive solution iff the Hammerstein operator has a positive eigenvalue, i.e. the Hammerstein equation

$$H_\alpha g = \lambda g, \quad f \in C^+[0,1] \tag{2.3}$$

has a positive solution in $\mathcal{M}_0$ for some $\lambda > 0$.

Proof. We define the linear operator W and the linear functional ω on the $C[0,1]$ by following equalities

$$(Wf)(t) = \int_0^1 K(t,u)f(u)du, \quad \omega(f) = \int_0^1 K(0,u)f(u)du.$$

Necessariness. Let $f_0 \in C_0^+[0,1]$ be a solution of the equation (2.2). We have

$$(Wf_0)(t) = \omega(f_0)\sqrt[\alpha]{f_0(t)}.$$

From this equality we get

$$(H_\alpha h)(t) = \lambda_0 h(t),$$

where $h(t) = \sqrt[\alpha]{f_0(t)}$ and $\lambda_0 = \omega(f_0) > 0$.

It is easy to see that $h \in \mathcal{M}_0$ and $h(t)$ is an eigenfunction of the Hammerstein's operator H_α, corresponding the positive eigenvalue λ_0.

Sufficiency. Let $h \in \mathcal{M}_0$ be an eigenfunction of the Hammerstein's operator H_α. Then there is a number $\lambda_0 > 0$ such that $H_\alpha h = \lambda_0 h$. From $h(0) = 1$ we get $\lambda_0 = (H_\alpha h)(0) = \omega(h^\alpha)$. Then

$$h(t) = \frac{(H_\alpha h)(t)}{\omega(h^\alpha)}.$$

From this equality we get $R_\alpha f_0 = f_0$ with $f_0 = h^\alpha \in C_0^+[0,1]$. This completes the proof. □

Theorem 2. *The equation (2.2) has at least one solution in $C_0^+[0,1]$.*

Let λ_0 be a positive eigenvalue of the Hammerstein operator $H_\alpha, \alpha > 1$. Then there exists $f_0 \in \mathcal{M}_0$ such that $H_\alpha f_0 = \lambda_0 f_0$. Take $\lambda \in (0, +\infty)$, $\lambda \neq \lambda_0$. Define function $h_0(t) \in C_0^+[0,1]$ by

$$h_0(t) = \sqrt[\alpha-1]{\frac{\lambda}{\lambda_0}} f_0(t), \quad t \in [0,1].$$

Then

$$H_\alpha h_0 = H_\alpha \left(\sqrt[\alpha-1]{\frac{\lambda}{\lambda_0}} f_0 \right) = \lambda h_0,$$

i.e. the number λ is an eigenvalue of Hammerstein operator H_α corresponding the eigenfunction $h_0(t)$. It can be easily checked: if the number $\lambda_0 > 0$ is eigenvalue of the operator $H_\alpha, \alpha > 1$, then an arbitrary positive number is eigenvalue of the operator H_α. Therefore we have

Lemma 2. *a)Let $\alpha > 1$.The equation $R_\alpha f = f$ has a nontrivial positive solution iff the Hammerstein equation $H_\alpha g = g$ has a nontrivial positive solution.*

Let $\alpha > 1$. Denote by $N_{fix.p}(H_\alpha)$ and $N_{fix.p}(R_\alpha)$ numbers of nontrivial positive solutions of the equations (2.1) and (2.2), respectively.

Theorem 3. *Let $\alpha > 1$. The equality $N_{fix.p}(H_\alpha) = N_{fix.p}(R_\alpha)$ is held.*

Denote

$$m = \min_{t,u\in[0,1]} K(t,u), \quad M_0 = \max_{u\in[0,1]} K(0,u),$$

$$M = \max_{t,u\in[0,1]} K(t,u), \quad m_0 = \min_{u\in[0,1]} K(0,u).$$

Theorem 4. *Let $\alpha > 1$. If the following inequality holds*

$$\left(\frac{M}{m_0}\right)^\alpha - \left(\frac{m}{M_0}\right)^\alpha < \frac{1}{\alpha}$$

then the homogenous Hammerstein equation (2.1) and the equation (2.2) has unique nontrivial positive solution.

Analogous theorem was proved for $\alpha = k \in \mathbb{N}, k \geq 2$ in [12] and proof of the Theorem 4 is analogously to its.

3. Existence finite positive solutions of homogeneous Hammerstein equation

In this section we'll show the existence of $n \in \mathbb{N}$ positive solutions of homogeneous integral equation of Hammerstein type (1.2).

For all $p, n \in \mathbb{N}$ we define following matrices:

$$\mathbf{A}_n^{(p)} = \left\{ \frac{1}{2(2p+i+j)-3} \left(\frac{1}{2}\right)^{2(2p+i+j-2)} \right\}_{i,j=\overline{1,n}}, \quad n, p \in \mathbb{N}. \tag{3.1}$$

$$\mathbf{B}[a_1, ..., a_n; b_1, ...b_n] = \left(\frac{1}{a_i + b_j} \right)_{i,j=\overline{1,n}}, \quad a_i, b_j > 0. \tag{3.2}$$

$$\mathbf{C}_n^{(p)} = B[4p, 4(p+1), ..., 2(p+n-1); 1, 5, ..., 4n-3]. \tag{3.3}$$

Lemma 3. *[24] Let $n \geq 2$. Then*

$$\det \mathbf{B}[a_1, ..., a_n; b_1, ..., b_n] = \frac{\prod_{1 \leq i < j \leq n}[(a_i - a_j)(b_i - b_j)]}{\prod_{i,j=1}^{n}(a_i + b_j)}$$

Corollary 1. $\det \mathbf{A}_n^{(p)} = \left(\frac{1}{2}\right)^{2n(2p+n-1)} \det \mathbf{C}_n^{(p)}$.

Proof. Let $i, j = \overline{1,n}$. We multiply by $2^{2(p+j-1)}$ the column j of the matrix $\mathbf{A}_n^{(p)}$ and then multiply by $2^{2(i-1)}$ the row i of the matrix obtained. As a result we get $\mathbf{C}_n^{(p)}$. □

Lemma 4. *Let $\mathbf{B}^{-1}[a_1, a_2, ..., a_n; b_1, b_2, ..., b_n] = \{\beta_{ij}\}_{i,j=\overline{1,n}}$ is an inverse matrix of $\mathbf{B}[a_1, a_2, ..., a_n; b_1, b_2, ..., b_n]$. Then*

$$\beta_{ji} = \frac{\prod_{s=1}^{n}(a_s + b_j)\prod_{s=1, s\neq i}^{n}(a_i + b_s)}{\prod_{s=1,s\neq j}^{n}(b_j - b_s)\prod_{s=1,s\neq i}^{n}(a_i - a_s)}$$

Proof. Subtracting the jth column of $\mathbf{B}[a_1, a_2, ..., a_n; b_1, b_2, ..., b_n]$ from every other column we get a following equality

$$\det \mathbf{B}[a_1, a_2, ..., a_n; b_1, b_2, ..., b_n] =$$

$$= \frac{\prod_{s=1,s\neq j}^{n}(b_j - b_s)}{\prod_{s=1}^{n}(a_s + b_j)} \begin{pmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_{j-1}} & 1 & \frac{1}{a_1+b_{j+1}} & \cdots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & \cdots & \frac{1}{a_2+b_{j-1}} & 1 & \frac{1}{a_2+b_{j+1}} & \cdots & \frac{1}{a_2+b_n} \\ & \cdots & & \cdots & & \cdots & \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_{j-1}} & 1 & \frac{1}{a_n+b_{j+1}} & \cdots & \frac{1}{a_n+b_n} \end{pmatrix}.$$

Now we subtract from the jth row the ith row for every $j \in \{1, 2..., i-1, i+1, ...n\}$. Then

$$\det \mathbf{B}[a_1, ..., a_n; b_1, ..., b_n] = \frac{\prod_{s=1,s\neq j}^{n}(b_j - b_s)\prod_{s=1,s\neq i}^{n}(a_i - a_s)}{\prod_{s=1}^{n}(a_s + b_j)\prod_{s=1,s\neq i}^{n}(a_i + b_s)} \times \det \mathbf{B}^{(i,j)}[a_1, ..., a_n; b_1, ..., b_n]$$

where $\mathbf{B}^{(j,i)}[a_1, ..., a_n; b_1, ..., b_n]$ is the cofactor of the element $\frac{1}{a_i+a_j}$ in $\mathbf{B}[a_1, ..., a_n; b_1, ..., b_n]$. Since

$$\beta_{ji} = \frac{\det \mathbf{B}^{(i,j)}[a_1, ..., a_n; b_1, ..., b_n]}{\det \mathbf{B}[a_1, ..., a_n; b_1, ..., b_n]}.$$

This completes the proof. □

Let be

$$(\mathbf{A}_n^{(p)})^{-1} = \{\alpha_{ij}\}_{i,j\in\overline{1,n}}.$$

Remark 1. *For each α_{ji} element of $(\mathbf{A}_n^{(p)})^{-1}$ the following equality holds*

$$\alpha_{ji} = 4^{2p+i+j-n+1} \cdot \frac{\prod_{s=1}^{n}(4p+2s+2j-3)\prod_{s=1,s\neq j}^{n}(4p+2s+2j-3)}{\prod_{s=1,s\neq j}^{n}(j-s)\prod_{s=1,s\neq i}^{n}(i-s)}$$

Proof. By Corollary 1 and Lemma 4 we get

$$\alpha_{ji} = 4^{2p+i+j} \cdot \frac{\det \mathbf{B}^{(i,j)}[4p, 4p+2, ...4p+2(n-1); 1, 3, ...2n-1]}{\det \mathbf{B}[4p, 4p+2, ...4p+2(n-1); 1, 3, ...2n-1]} =$$
$$= 4^{2p+i+j} \cdot \frac{\prod_{s=1}^{n}(4p+2s+2j-3)\prod_{s=1,s\neq i}^{n}(4p+2s+2i-3)}{\prod_{s=1,s\neq j}^{n}(2j-2s)\prod_{s=1,s\neq i}^{n}(2i-2s)}.$$

□

Denote

$$\varphi_{(s,n,p)}(u) = \alpha_{s1}u^{2p-1} + ... + \alpha_{sn}u^{2(n+p)-3}, \quad s, n, p \in \mathbb{N}, \ u \in [0,1].$$

$$K_{(n,p)}(t, u; k) = 1 + \sum_{s=1}^{n}\left(\sqrt[k]{1+t^{2(p+s)-1}} - 1\right)\varphi_{(s,n,p)}(u), \quad k \in \mathbb{N}, k \geq 2, \ t, u \in [0,1].$$

Remark 2. *For the given $k \in \mathbb{N}, k \geq 2$ the following inequality holds*

$$K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) \leq K_{(n,1)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right), \quad (t,u) \in [0,1]^2, n, p \in \mathbb{N}.$$

Put

$$\zeta_0(n) = \frac{64}{9} \cdot \frac{4^n-1}{4n+1}\left(\frac{(4n+1)!!}{(n-1)!(2n+1)!!}\right)^2.$$

Lemma 5. *Let $n \in \mathbb{N}$. If $k \geq \zeta_0(n)$ then the following inequality holds*

$$K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) > 0, \ (t,u) \in [0,1]^2, p \in \mathbb{N}.$$

Proof. For $p = 1$ from Remark 1 we have

$$\alpha_{ij} = 4^{i+j-n+3}\frac{\prod_{s=1}^{n}(2i+2s+1)\prod_{s=1,s\neq j}^{n}(2j+2s+1)}{\prod_{s=1,s\neq i}^{n}(i-s)\prod_{s=1,s\neq j}^{n}(j-s)}.$$

Then

$$\left|\frac{\alpha_{i,j+1}}{\alpha_{i,j}}\right| = \frac{4(4j+1)(2j+2n+3)}{(2j+3)(4j+5)}, \ i = \overline{1,n}, \ j = \overline{1,n-1}$$

and

$$\left|\frac{\alpha_{i+1,j}}{\alpha_{i,j}}\right| = \frac{4(n-i)(2i+2n+3)}{i(2i+3)}, \ i=\overline{1,n}, \ j=\overline{1,n-1}.$$

From above one has $\max_{i,j=\overline{1,n}}|\alpha_{ij}| = |\alpha_{nn}|$. By Remark 1 we can take

$$K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) \geq 1 - \frac{2}{3}\max_{i,j=\overline{1,n}}|\alpha_{ij}| \sum_{s=1}^{n}\left(\sqrt[k]{1+\left(\frac{1}{2}\right)^{2s+1}}-1\right) \geq$$

$$\geq 1 - \frac{2|a_{nn}|}{3k}\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{2s+1} \geq 1-(4^n-1)\cdot\frac{64(2n+3)^2(2n+5)^2...(4n-1)^2(4n+1)}{9k\left((n-1)!\right)^2}.$$

Since $k \geq \zeta_0(n)$ one get $K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) > 0$. This completes the proof. □

Proposition 1. *Let $n \in \mathbb{N}$. If $k \geq \zeta_0(n)$ then the Hammerstein's nonlinear operator H_k with the kernel $K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right)$ $(p \in \mathbb{N})$ has at least n positive fixed points.*

Proof. Let $f_j(u) = \sqrt[k]{1+u^{2(p+j)-1}}, \ j=\overline{1,n}$ and $u_1 = u-\frac{1}{2}, t_1 = t-\frac{1}{2}$. Put $g_j(t) = f_j(t-\frac{1}{2})$. We are showing functions $g_j(t)$ are fixed points of the Hammerstein operator H_k with the kernel $K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right)$:

$$\int_0^1 K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) g_j^k(u)du =$$

$$= \int_0^1 K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) f_j^k\left(u-\frac{1}{2}\right)du = \int_{-\frac{1}{2}}^{\frac{1}{2}} K_{(n,p)}\left(t_1, u_1; k\right) f_j^k(u_1)du_1 =$$

$$\int_{\frac{1}{2}}^{-\frac{1}{2}}\left[1+\sum_{s=1}^{n}\left(\sqrt[k]{1+t_1^{2(p+s)-1}}-1\right)\varphi_{(s,n,p)}(u_1)\right]\left(1+u_1^{2(p+j)-1}\right)du_1 =$$

$$1+\sum_{s=1}^{n}\left(\sqrt[k]{1+t_1^{2(p+s)-1}}-1\right)\int_{\frac{1}{2}}^{\frac{1}{2}}\left(\alpha_{s1}u_1^{4(p-1)+2s+2j}+...+\alpha_{sn}u_1^{4p+2(s+j+n)-6}\right)du_1 =$$

$$1+\sum_{s=1}^{n}\left(\sqrt[k]{1+t_1^{2(p+s)-1}}-1\right)(\alpha_{sj}\beta_{s1}+...+\alpha_{nj}\beta_{sn}) = \sqrt[k]{1+t_1^{2(p+j)-1}}.$$

Hence

$$\int_0^1 K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) g_j^k(u)du = g_j(t), \quad j \in \{1,2,...,n\}.$$

□

Theorem 5. *For each $n \in \mathbb{N}$ there exists $\vartheta > 1$ and a positive continuous kernel $K(t,u)$ such that Hammerstein integral equation (1.2) has n positive solutions.*

4. Gibbs measures for models on Cayley tree Γ^k

In this section we study Gibbs measures for models on Cayley tree. You may be acquaint with definitions and properties of Gibbs measures in books [14], [23], [25]. A Cayley tree (Bethe lattice) Γ^k of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L that of edges (arcs). Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y\rangle$. A collection of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, ...\langle x_{d-1}, y\rangle$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n,$$

the set of *direct successors* of x.

Consider models where the spin takes values in the set $[0, 1]$, and is assigned to the vertexes of the tree. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0, 1]$. Denote $\Omega_A = [0, 1]^A$ the set of all configurations on A and $\Omega = [0, 1]^V$. The Hamiltonian on Γ^k of the model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi\left(\sigma(x), \sigma(y)\right), \quad \sigma \in \Omega \tag{4.1}$$

where $J \in R \setminus \{0\}$ and $\xi : (u, v) \in [0, 1]^2 \to \xi_{u,v} \in \mathbb{R}$ is a given bounded, measurable function.

Let λ be the Lebesgue measure on $[0, 1]$. On the set of all configurations on A the a priori measure λ_A is introduced as the $|A|$ fold product of the measure λ. Here and further on $|A|$ denotes the cardinality of A. We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega = [0, 1]^V$ generated by the measurable cylinder subsets.

Let $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]) \in \mathbb{R}^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right). \tag{4.2}$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda_{V_n}(\widetilde{\sigma}_n), \tag{4.3}$$

where $\beta = T^{-1}, T > 0-$ temperature. The probability distributions $\mu^{(n)}$ are compatible [13] if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)\lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{4.4}$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. In this case there exists [13] a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$. The measure μ is called *splitting Gibbs measure* corresponding to Hamiltonian (4.1) and function $x \mapsto h_x$, $x \neq x^0$.

The following statement describes conditions on h_x guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

Proposition 2. *[13] The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (4.2) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$f(t,x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{t,u}) f(u,y)du}{\int_0^1 \exp(J\beta\xi_{0,u}) f(u,y)du}. \tag{4.5}$$

Here, and below $f(t,x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0,1]$ and $du = \lambda(du)$ is the Lebesgue measure.

We consider ξ_{tu} as a continuous function and we are going to solve equation (4.5) in the class of $translation - invariant$ functions $f(t,x)$ (i.e. $f(t,x) = f(t)$ for all $x \in \Gamma^k \setminus \{x_0\}$) and we'll show that there exists a finite number of $translation - invariant$ Gibbs measures for model (4.1).

Then for *translation-invariant* functions the equation (4.5) can be written as

$$(R_k f)(t) = f(t), \ k \in \mathbb{N} \tag{4.6}$$

where $K(t,u) = Q(t,u) = \exp(J\beta\xi_{tu})$, $f(t) \in C_0^+[0,1]$, $t, u \in [0,1]$ (see [13], [12]).

Consequently, for each $k \in \mathbb{N}, k \geq 2$ Hammerstein integral equation corresponding to the equation (4.6) has the following form:

$$\int_0^1 Q(t,u) f^k(u)du = f(t). \tag{4.7}$$

By Theorem 3 and Propositions 1, 2 we'll get following Theorem.

Theorem 6. *Let $n \in \mathbb{N}$. If $k \geq \zeta_0(n)$ then the model*

$$H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} ln\left(K_{(n,p)}\left(\sigma(x) - \frac{1}{2}, \sigma(y) - \frac{1}{2}; k\right)\right), \quad \sigma \in \Omega, (p \in \mathbb{N})$$

on the Cayley tree Γ^k has at least n translation-invariant Gibbs measures.

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Yu. Kh. Eshkabilov, National University of Uzbekistan, Tashkent, Uzbekistan.
E-mail address: yusup62@mail.ru

F.H.Haydarov, National University of Uzbekistan, Tashkent, Uzbekistan.
E-mail address: haydarov_ imc@mail.ru.